General Analytical Solutions of Static Green's Functions for Shielded and Open Arbitrarily Multilayered Media

Keren Li, *Member, IEEE,* Kazuhiko Atsuki, *Member, IEEE,* and Tsuyoshi Hasegawa, *Student Member, IEEE*

***Abstract*—In this paper, we present for the first time general analytical solutions of the static Green's functions for shielded and open arbitrarily multilayered media. The analytical formulas for the static Green's functions, which are expressed in the form of the Fourier series or the Fourier integrals, have simple form and are applicable to arbitrary number of the dielectric layers. The derivation of the formulas is primarily based on a technique by which a recurrence relation between L layers and $L+1$ layers is developed. Green's functions for a three-layered dielectric structure are given as an example of the general formulas. These general analytical solutions will provide a new and efficient tool to the analysis of the multilayered medium structures.**

I. Introduction

To FIND a Green's function is the first and the most important step in solving the integral equations formulated by the boundary integral equation techniques such as the boundary element method (BEM) [1]–[4] or the partial-boundary element method (p-BEM) [5] and the spectral-domain approach (SDA) [6]–[10]. For free space or for an unbounded homogenous space, the Green's function is easily obtained and is of a simple closed form for both static and time-harmonic fields. However, quite often in practice it is almost as difficult to find a solution for the Green's function as it is to solve the original boundary-value problem.

Fig. 1 shows a two-dimensional (2-D) multilayered medium structure shielded with rectangular conductor walls. The multilayered medium structure has been recently developed and introduced in many microwave circuits, especially in monolithic microwave integrated circuits (MMIC's), to provide high-cost performance as commercial products [11]. The efficient analysis of such multilayered medium structures usually requires Green's functions incorporating partially or all the boundary conditions in the multilayered structures [3]–[10]. Such Green's functions, however, are no longer easy to obtain.

One of the most conventional Green's function for the structure shown in Fig. 1 is given in the summation of each Green's function created by the image charges in the multilayered media. The other is expanded in the Fourier series, i.e., in spectral domain [1]. For a one-layered dielectric structure, the Green's function is expressed as an infinite summation over image charges, whereas for two or more

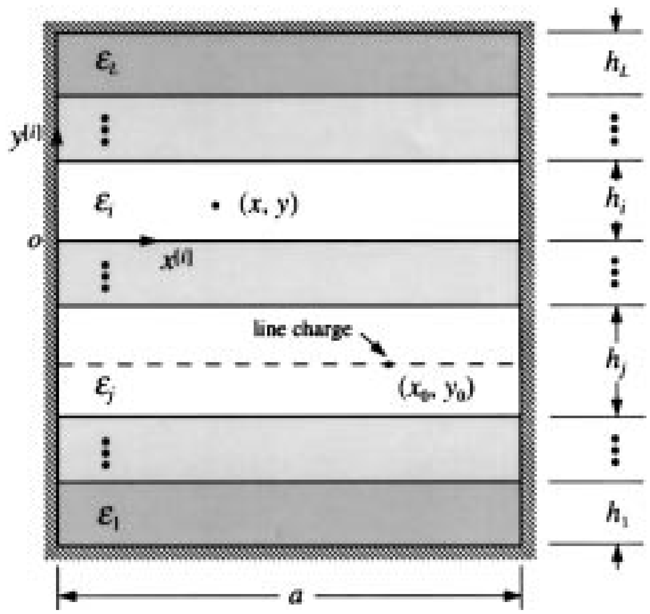

Fig. 1. A 2-D multilayered medium structure shielded with conductor walls.

layered media the Green's function is of a form of double or multi-infinite summations, which may make the computation very complicated and inefficient [12]. On the other hand, with the Fourier series expanded in the spectral domain, the Fourier series is only one infinite summation regardless of the number of the dielectric layers. To determine such Green's function for the structure shown in Fig, 1, incorporating all boundary conditions on the interfaces of the dielectric layers and on the grounded conductor walls, it is necessary to relate the Fourier coefficients in each layer to the boundary conditions. When the number of dielectric layers is large (for example, more than four layers) this procedure is exhaustive work. For such structures with large number of layers, there is an effective technique to derive the Fourier series in an semiautomatic procedure, where the dielectric layer is simulated to a transverse transmission line and then the transmission line theory can be employed to relate the Fourier coefficients between two adjacent dielectric layers [7], [8], [13], [14]. However, even with this procedure the obtained Green's function is usually not given by a final expression for the arbitrary number of layers, so that it must be derived each time for a given number of layers. This may take a lot of derivation time. Especially, for the shielded structure as

Manuscript received August 1, 1994; revised September 23, 1996.
The authors are with the Department of Electronic Engineering, the University of Electro-Communications, Chofu-shi, Tokyo 182, Japan.
Publisher Item Identifier S 0018-9480(97)00259-7.

shown in Fig. 1, the Fourier coefficients must be calculated for each Fourier term—not analytically, but numerically—as is the Green's function. For this reason, there is a need to develop an analytical Green's function for the multilayered medium structure. Furthermore, an analytical expression for the Green's function to be found may help us to treat the singularity occurred in the Green's function and its boundary integrals and to improve the convergence of the Green's function in numerical calculation.

In this paper, first we give a general analytical solution of the static Green's function for the shielded structure with multilayered media (as shown in Fig. 1), and then we give the detail derivation. The general analytical formula for the Green's function is expressed in the Fourier series or the Fourier integrals (Sections II and IV). The formula has simple form and is applicable to an arbitrary number of multilayers. The derivation of the formula is primarily based on a technique by which a recurrence relation between L layers and $L+1$ layers is developed. The details of the derivation are described in Section III. The analytical Green's function for an open

multilayered media is also derived and given in Section IV. Green's functions for a three-layered dielectric structure are provided in Section V as an example of the general formula.

II. Analytical Green's Functions for an Arbitrarily Multilayered Media with Shielded Conductor Walls

As illustrated in Fig. 1, the arbitrarily multilayered medium structure consists of L isotropic dielectric layers with electric parameters $\varepsilon_i (i = 1, 2, \cdots, L)$ and perfectly conducting shielded conductor walls. The source point and the observation point are placed in jth and ith layer and denoted by (x_0, y_0) and (x, y), respectively. The source is a line charge for the 2-D problem. Coordinate systems for analysis are built in each local layer, as shown in Fig. 2(a), for convenience of analysis. Hence, the y_0 and y in such local coordinate systems should have the values in the region $0 \leq y_0 \leq h_j$ and $0 \leq y \leq h_i$, respectively. In this paper, we define a layer as "the source layer" when the line charge exists in that layer or as "the nonsource layer" otherwise.

For an electrostatic problem or an analysis under the quasi-TEM wave approximation, the Green's function under consideration, as shown in Fig. 2(a), satisfies following Laplace equation and Poisson equation [1], [2]:

$$\nabla^2 G_{ij}(x, y|x_0, y_0) = 0, \qquad i \neq j \tag{1}$$

$$\nabla^2 G_{jj}(x, y|x_0, y_0) = -\frac{1}{\varepsilon_j}\delta(x - x_0, y - y_0) \tag{2}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is a Laplacian operator.

Setting the potentials on the shielded conductor walls at zero volt, the Green's functions in ith layer can be expressed in the form of Fourier series as

$$G_{ij}(x, y|x_0, y_0) = \sum_{n=1}^{\infty} (A_n^{(i)} \cosh \alpha_n y + B_n^{(i)} \sinh \alpha_n y) \sin \alpha_n x \tag{3}$$

where $\alpha_n = n\pi/a, n = 1, 2, \cdots, \infty$ and $A_n^{(i)}$ and $B_n^{(i)}$ are the unknown Fourier coefficients for the ith layer.

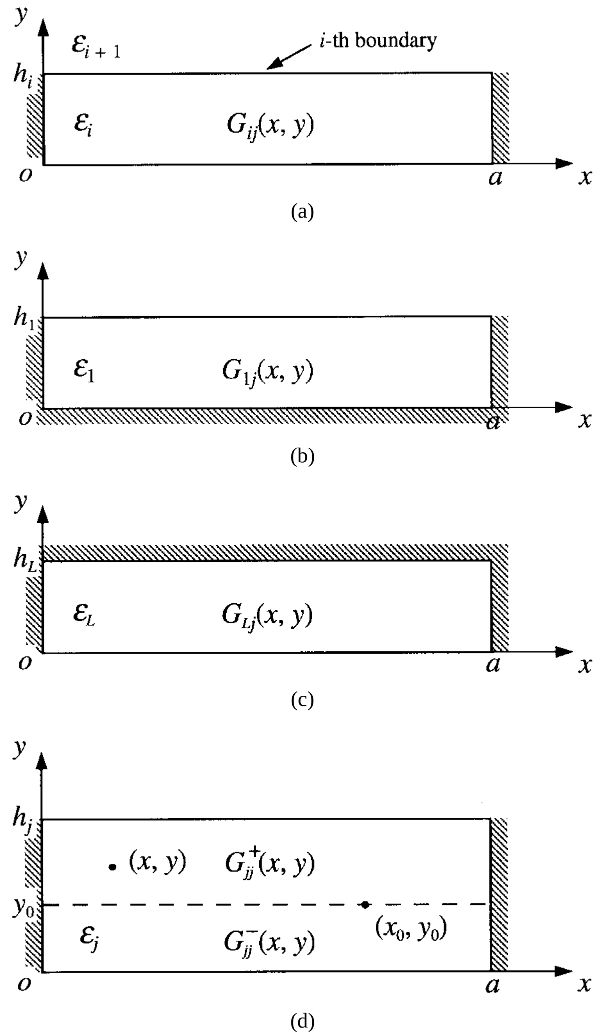

Fig. 2. Local coordinates and Green's functions in different layers. (a) Local coordinates for ith layer. (b) Green's function in the first layer. (c) Green's function in the last layer. (d) Green's function in jth layer where the line charge source exists at point (x_0, y_0).

The Green's function given in (3) at arbitrary observation point and source point in arbitrary layer can be derived analytically and given by following formula:

$$G_{ij}(x, y|x_0, y_0) = \sum_{n=1}^{\infty} \frac{1}{\alpha_n a} \sin \alpha_n x \sin \alpha_n x_0 \cdot \frac{2}{\Delta_n} \begin{cases} \Delta_n^{i+}(\overline{h_i} - \overline{y})\Delta_n^{j-}(\overline{y_0}), \\ \qquad i > j \text{ or } i = j, y \geq y_0 \\ \Delta_n^{i-}(\overline{y})\Delta_n^{j+}(\overline{h_j} - \overline{y_0}), \\ \qquad i < j \text{ or } i = j, y \leq y_0 \end{cases} \tag{4}$$

where

$$\overline{h_k} = \alpha_n h_k, \qquad k = 1, 2, \cdots, L \tag{5a}$$

$$\overline{y} = \alpha_n y, \quad \overline{y_0} = \alpha_n y_0 \tag{5b}$$

$$\Delta_n = \Delta(\overline{h_1}, \overline{h_2}, \cdots, \overline{h_L}) \tag{6a}$$

$$\Delta_n^{i+}(\overline{y}) = \Delta(0, 0, \cdots, \overline{y}, \overline{h_{i+1}}, \cdots, \overline{h_L}) \tag{6b}$$

$$\Delta_n^{i-}(\overline{y}) = \Delta(\overline{h_1}, \overline{h_2}, \cdots, \overline{h_{i-1}}, \overline{y}, 0, \cdots, 0) \tag{6c}$$

and $\Delta(z_1, z_2, \cdots, z_L)$ is a generating function given by

$$\Delta(z_1, z_2, \cdots, z_L) = \prod_{k=1}^{L} \frac{1}{\varepsilon_k} \cdot \prod_{k=1}^{L-1} \left(\varepsilon_k \frac{\partial}{\partial z_k} + \varepsilon_{k+1} \frac{\partial}{\partial z_{k+1}} \right) \cdot \prod_{k=1}^{L} \sinh z_k. \tag{7}$$

To use this general formula, there is a simple rule by which when $y \geq y_0$, we select the function with the "+" sign for the observation point y and its argument of the function is $(\overline{h_i} - \overline{y})$ and the function for the source point y_0 is with the "−" sign and its argument is only y_0; when $y_0 \geq y$, we only need to exchange y_0 and y and the corresponding layer number. The signs "+" and "−" correspond to the relative height of the observation point and of the source point; the higher uses "+," and the lower uses "−."

The details of the derivation of this general analytical solution will be given below.

III. Derivation of the Analytical Green's Functions for the Shielded Multilayered Media

A. Expressions for the Fourier Coefficients

Applying the boundary conditions on the upper boundary of ith layer, as shown in Fig. 2(a), to the Green's function in (3), we have

$$\left.\begin{aligned} G_{ij}|_{y=h_i} &= G_{i+1,j}|_{y=0} \\ \varepsilon_i \left.\frac{\partial G_{ij}}{\partial y}\right|_{y=h_i} &= \varepsilon_{i+1} \left.\frac{\partial G_{i+1,j}}{\partial y}\right|_{y=0} \end{aligned}\right\} \quad \text{on } i\text{th boundary}. \tag{8}$$

From these two equations we get a relation equation between the two sets of Fourier coefficients as follows:

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix}^{(i+1)} = \boldsymbol{F}_i \begin{bmatrix} A_n \\ B_n \end{bmatrix}^{(i)} \quad \text{or} \quad \begin{bmatrix} A_n \\ B_n \end{bmatrix}^{(i)} = \boldsymbol{F}_i^{-1} \begin{bmatrix} A_n \\ B_n \end{bmatrix}^{(i+1)} \tag{9}$$

where the coefficient matrix in above equation is given by

$$\boldsymbol{F}_i = \frac{1}{\varepsilon_{i+1}} \begin{bmatrix} \varepsilon_{i+1} \cosh \overline{h_i} & \varepsilon_{i+1} \sinh \overline{h_i} \\ \varepsilon_i \sinh \overline{h_i} & \varepsilon_i \cosh \overline{h_i} \end{bmatrix} \tag{10}$$

and the inverse matrix is

$$\boldsymbol{F}_i^{-1} = \frac{1}{\varepsilon_i} \begin{bmatrix} \varepsilon_i \cosh \overline{h_i} & -\varepsilon_{i+1} \sinh \overline{h_i} \\ -\varepsilon_i \sinh \overline{h_i} & \varepsilon_{i+1} \cosh \overline{h_i} \end{bmatrix}. \tag{11}$$

The symbols with over bar in above matrices are defined in (5).

Referring to the expression in (3), the Green's functions in the first and last layers, as shown in Fig. 2(b) and 2(c), are

$$G_{1j,n} \propto \sinh \alpha_n y \quad \text{and} \quad G_{Lj,n} \propto \sinh \alpha_n (h_L - y). \tag{12}$$

Therefore, we can define that

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix}^{(1)} = \boldsymbol{F}_0 C_n^-, \quad \text{and} \quad \begin{bmatrix} A_n \\ B_n \end{bmatrix}^{(L)} = \boldsymbol{F}_L^{-1} C_n^+ \tag{13}$$

where

$$\boldsymbol{F}_0 = \frac{1}{\varepsilon_1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{F}_L^{-1} = \frac{1}{\varepsilon_L} \begin{bmatrix} \sinh \overline{h_L} \\ -\cosh \overline{h_L} \end{bmatrix} \tag{14}$$

and C_n^- and C_n^+ in (13) are two unknown coefficients for the first and last layers, i.e., the bottom and top layers.

Substituting (13) and (14) into (9), we can get the relation equation between the Fourier coefficients for ith layer in (3) and the unknown coefficients C_n^- and C_n^+ for the bottom and top layers as follows:

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix}^{(i)} = \boldsymbol{F}_i^+ C_n^+ \quad \text{or} \quad \begin{bmatrix} A_n \\ B_n \end{bmatrix}^{(i)} = \boldsymbol{F}_i^- C_n^- \tag{15}$$

where the matrices $\boldsymbol{F}_i^+$ and $\boldsymbol{F}_i^-$ are defined as follows:

$$\boldsymbol{F}_i^+ = \prod_{k=i}^{L} \boldsymbol{F}_k^{-1} = \begin{bmatrix} a_n^+ \\ b_n^+ \end{bmatrix}_{(i)}^{(L)} \tag{16a}$$

$$\boldsymbol{F}_i^- = \prod_{k=i}^{1} \boldsymbol{F}_{k-1} = \begin{bmatrix} a_n^- \\ b_n^- \end{bmatrix}_{(1)}^{(i)}. \tag{16b}$$

The coefficients a_n^+, b_n^+, a_n^-, and b_n^- in above equations are introduced for convenience of the later derivation.

B. Green's Function in the Source Layer

In the source layer, the Green's function satisfies Poisson's equation (2). However, after dividing the layer into two regions by adding a boundary at $y = y_o$, as shown in Fig. 2(d), the Green's functions in the two regions then satisfy Laplace equations and can be expanded as two Fourier series, similar to the series shown in (3) as

$$G_{jj}^{\pm}(x, y|x_0, y_0) = \sum_{n=1}^{\infty} (A_n^{\pm} \cosh \alpha_n y + B_n^{\pm} \sinh \alpha_n y) \sin \alpha_n x \qquad y \neq y_0 \tag{17}$$

where the plus and minus signs are responding to the upper and lower regions shown in Fig. 2(d), respectively, and the Fourier coefficients in (17) can be found from the (16) and given by

$$\begin{bmatrix} A_n^{\pm} \\ B_n^{\pm} \end{bmatrix} = \boldsymbol{F}_j^{\pm} C_n^{\pm} = \begin{bmatrix} a_n^{\pm} \\ b_n^{\pm} \end{bmatrix} C_n^{\pm}. \tag{18}$$

Note that the superscripts $(L), (j)$ and subscripts $(j), (1)$ of the coefficients a_n^+, b_n^+, a_n^-, and b_n^- are omitted in above equation for simplicity of expression.

To find the coefficients C_n^- and C_n^+ in (18), let us consider the boundary conditions at $y = y_o$, which are given by

$$G_{jj}^+|_{y=y_0} = G_{jj}^-|_{y=y_0} \tag{19a}$$

$$-\varepsilon_j \left.\left(\frac{\partial G_{jj}^+}{\partial y} - \frac{\partial G_{jj}^-}{\partial y} \right)\right|_{y=y_0} = \delta(x - x_0). \tag{19b}$$

Substituting (17) and (18) into (19) gives solutions

$$C_n^{\pm} = \frac{\Delta_{n1}^{\mp}}{\Delta_n} s_n \tag{20}$$

where

$$\Delta_{n1}^{\pm}(y_0) = a_n^{\pm}\cosh\alpha_n y_0 + b_n^{\pm}\sinh\alpha_n y_0 \tag{21}$$

$$s_n = \frac{1}{\alpha_n}\left(\frac{2}{a}\int_0^a \delta(x-x_0)\sin\alpha_n x\right) = \frac{2}{\alpha_n a}\sin\alpha_n x_0 \tag{22}$$

$$\Delta_n = \varepsilon_j(a_n^+ b_n^- - a_n^- b_n^+). \tag{23}$$

Then we have the Green's functions in the source layer as follows:

$$G_{jj}^{\pm}(x,y|x_0,y_0) = \sum_{n=1}^{\infty}\frac{2}{\alpha_n a}\frac{\Delta_{n1}^{\pm}(y)\Delta_{n1}^{\mp}(y_0)}{\Delta_n}\sin\alpha_n x\sin\alpha_n x_0. \tag{24}$$

C. Derivation of the Generating Function $\Delta(z_1, z_2, \cdots, z_L)$

To derive an analytical Green's function for arbitrary number of layers, let us rewrite the Δ_n in (23) for the case of L layers as

$$\Delta_n^L = \varepsilon_j(a_n^{L+} b_n^- - a_n^- b_n^{L+}) \tag{25}$$

where, from (16a), we have

$$\begin{bmatrix} a_n^+ \\ b_n^+ \end{bmatrix}^{(L)} = \prod_{k=j}^{L}\boldsymbol{F}_k^{-1} = \left(\prod_{k=j}^{L-1}\boldsymbol{F}_k^{-1}\right)\frac{1}{\varepsilon_L}\begin{bmatrix}\sinh\overline{h_L} \\ -\cosh\overline{h_L}\end{bmatrix}.$$

The subscripts in above two equations are omitted for simplicity.

Letting

$$\frac{1}{\varepsilon_L}\left(\prod_{k=j}^{L-1}\boldsymbol{F}_k^{-1}\right) = \begin{bmatrix} d_{11} & -d_{12} \\ -d_{21} & d_{22}\end{bmatrix} \tag{26}$$

we have

$$\begin{bmatrix} a_n^+ \\ b_n^+ \end{bmatrix}^{(L)} = \begin{bmatrix} d_{11}\sinh\overline{h_L} + d_{12}\cosh\overline{h_L} \\ -(d_{21}\sinh\overline{h_L} + d_{22}\cosh\overline{h_L})\end{bmatrix}. \tag{27}$$

Now, we consider the case of $L+1$ layers. The $(L+1)$th layer is added on the top of the Lth layer. Obviously, from (23) we have

$$\Delta_n^{L+1} = \varepsilon_j(a_n^{(L+1)+} b_n^- - a_n^- b_n^{(L+1)+}) \tag{28}$$

and from (16a)

$$\begin{aligned}\begin{bmatrix} a_n^+ \\ b_n^+ \end{bmatrix}^{(L+1)} &= \prod_{k=j}^{L+1}\boldsymbol{F}_k^{-1} \\ &= \frac{1}{\varepsilon_{L+1}}\frac{1}{\varepsilon_L}\left(\prod_{k=j}^{L-1}\boldsymbol{F}_k^{-1}\right)\cdot\varepsilon_L\boldsymbol{F}_L^{-1}\cdot\varepsilon_{L+1}\boldsymbol{F}_{L+1}^{-1}.\end{aligned}$$

Since

$$\varepsilon_L\boldsymbol{F}_L^{-1}\cdot\varepsilon_{L+1}\boldsymbol{F}_{L+1}^{-1} = \begin{bmatrix}\varepsilon_L\cosh\overline{h_L}\sinh\overline{h_{L+1}} \\ \quad +\varepsilon_{L+1}\sinh\overline{h_L}\cosh\overline{h_{L+1}} \\ -(\varepsilon_L\sinh\overline{h_L}\sinh\overline{h_{L+1}} \\ \quad +\varepsilon_{L+1}\cosh\overline{h_L}\cosh\overline{h_{L+1}})\end{bmatrix}$$

then, from (26) and (27), we get

$$\begin{aligned}\begin{bmatrix} a_n^+ \\ b_n^+ \end{bmatrix}^{(L+1)} &= \frac{1}{\varepsilon_{L+1}}\begin{bmatrix} d_{11} & -d_{12} \\ -d_{21} & d_{22}\end{bmatrix}\varepsilon_L\boldsymbol{F}_L^{-1}\cdot\varepsilon_{L+1}\boldsymbol{F}_{L+1}^{-1} \\ &= \frac{1}{\varepsilon_{L+1}}\begin{bmatrix}\left(\varepsilon_L\frac{\partial}{\partial\overline{h_L}} + \varepsilon_{L+1}\frac{\partial}{\partial\overline{h_{L+1}}}\right)a_n^{L+} \\ \cdot\sinh\overline{h_{L+1}} \\ \left(\varepsilon_L\frac{\partial}{\partial\overline{h_L}} + \varepsilon_{L+1}\frac{\partial}{\partial\overline{h_{L+1}}}\right)b_n^{L+} \\ \cdot\sinh\overline{h_{L+1}}\end{bmatrix}.\end{aligned}$$

Then, from (28), we have

$$\frac{1}{\varepsilon_j}\Delta_n^{L+1} = \frac{1}{\varepsilon_{L+1}}\left(\varepsilon_L\frac{\partial}{\partial\overline{h_L}} + \varepsilon_{L+1}\frac{\partial}{\partial\overline{h_{L+1}}}\right)\cdot(a_n^{L+}b_n^- - a_n^- b_n^{L+})\sinh\overline{h_{L+1}}. \tag{29}$$

Relating (25) to this equation gives a recurrence relation between Δ_n^{L+1} and Δ_n^L as

$$\Delta_n^{L+1} = \frac{1}{\varepsilon_{L+1}}\left(\varepsilon_L\frac{\partial}{\partial\overline{h_L}} + \varepsilon_{L+1}\frac{\partial}{\partial\overline{h_{L+1}}}\right)\Delta_n^L\sinh\overline{h_{L+1}}. \tag{30}$$

Since for the structure with only one layer we have

$$\Delta_n^1 = \frac{1}{\varepsilon_1}\sinh\overline{h_1}$$

we can then get a generating formula from the recurrent (30) for Δ_n^L as

$$\Delta_n^L = \prod_{k=1}^{L}\frac{1}{\varepsilon_k}\cdot\prod_{k=1}^{L-1}\left(\varepsilon_k\frac{\partial}{\partial\overline{h_k}} + \varepsilon_{k+1}\frac{\partial}{\partial\overline{h_{k+1}}}\right)\cdot\prod_{k=1}^{L}\sinh\overline{h_k}. \tag{31}$$

Using a set of generalized symbols $z_k, k = 1, 2, \cdots, L$,we can rewrite the generating formula as shown in (7) and have

$$\Delta_n = \varepsilon_j(a_n^+ b_n^- - a_n^- b_n^+) = \Delta(\overline{h_1}, \overline{h_2}, \cdots, \overline{h_L}). \tag{32}$$

In this paper, the function $\Delta(z_1, z_2, \cdots, z_L)$ defined in (7) is called the generating function. This function is independent on where the source layer and the observation layer, i.e., jth and ith layers exist since we have never specified these layers in above derivation procedure.

D. Derivation of the Analytical Green's Functions in an Arbitrary Layer

In this section, we try to express the functions $\Delta_{n1}^+(y)$ and $\Delta_{n1}^-(y_0)$ defined in (21) in a more sophisticated way and relate these two functions to the generating function $\Delta(z_1, z_2, \cdots, z_L)$.

In the Source Layer $i = j$: We rewrite $\Delta_{n1}^+(y)$ from (21) and (16a) as follows:

$$\Delta_{n1}^+(y) = [\cosh\overline{y}, \sinh\overline{y}]\begin{bmatrix} a_n^+ \\ b_n^+ \end{bmatrix}_{(j)} \tag{33}$$

$$\begin{bmatrix} a_n^+ \\ b_n^+ \end{bmatrix}_{(j)} = \frac{1}{\varepsilon_j}\begin{bmatrix}\varepsilon_j\cosh\overline{h_j} & -\varepsilon_{j+1}\sinh\overline{h_j} \\ -\varepsilon_j\sinh\overline{h_j} & \varepsilon_{j+1}\cosh\overline{h_j}\end{bmatrix}\begin{bmatrix} a_n^+ \\ b_n^+ \end{bmatrix}_{(j+1)} \tag{34}$$

then substituting (34) into (33) gives

$$\Delta_{n1}^{+}(y) = \frac{1}{\varepsilon_j}[\varepsilon_j \cosh(\overline{h_j} - \overline{y}) - \varepsilon_{j+1} \sinh(\overline{h_j} - \overline{y})] \begin{bmatrix} a_n^+ \\ b_n^+ \end{bmatrix}_{(j+1)}. \quad (35)$$

Referring this expression to (34) again, we discover that

$$\Delta_{n1}^{+}(y) = a_n^{j+}(\overline{h_j} - \overline{y}) = a_n^{+}(\overline{h_j} - \overline{y}). \quad (36)$$

Note that the superscripts in above equations are omitted for simplicity.

On the other hand, letting

$$\overline{h_k} = 0, \qquad k = 1, 2, \cdots, j-1$$

and substituting these terms into (10) gives

$$\boldsymbol{F}'_k = \frac{1}{\varepsilon_{k+1}} \begin{bmatrix} \varepsilon_{k+1} & 0 \\ 0 & \varepsilon_k \end{bmatrix}$$

then

$$\begin{bmatrix} a_n'^- \\ b_n'^- \end{bmatrix} = \prod_{k=j}^{1} \boldsymbol{F}'_{k-1} = \frac{1}{\varepsilon_j} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, from (23), we have

$$\Delta(0, 0, \cdots, \overline{h_j}, \overline{h_{j+1}}, \cdots, \overline{h_L}) = \varepsilon_j (a_n^+ b_n'^- - a_n'^- b_n^+) = a_n^+(\overline{h_j}). \quad (37)$$

Then, obviously, we can get

$$\Delta_{n1}^{+}(y) = \Delta(0, 0, \cdots, \overline{h_j} - \overline{y}, \overline{h_{j+1}}, \cdots, \overline{h_L}). \quad (38)$$

Similarly, we can get $\Delta_{n1}^{-}(y_0)$ as follows:

$$\Delta_{n1}^{-}(y_0) = \Delta(\overline{h_1}, \overline{h_2}, \cdots, \overline{h_{j-1}}, \overline{y_0}, 0), \cdots, 0). \quad (39)$$

By defining two functions given in (6) for generality and using the symmetry of the Green's function, we can rewrite the Green's function in (24) as follows:

$$G_{jj}(x, y|x_0, y_0) = \sum_{n=1}^{\infty} \frac{1}{\alpha_n a} \sin \alpha_n x \sin \alpha_n x_0 \cdot \frac{2}{\Delta_n} \begin{cases} \Delta_n^{i+}(\overline{h_i} - \overline{y})\Delta_n^{j-}(\overline{y_0}), & y \ge y_0 \\ \Delta_n^{i-}(\overline{y})\Delta_n^{j+}(\overline{h_j} - \overline{y_0}), & y \le y_0. \end{cases} \quad (40)$$

In the Nonsource Layer $i \neq j$: Here, we can only consider the case $i > j$ for the symmetry of the Green's function. When $i > j$, the Fourier coefficients in ith layer are given by

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix}^{(i)} = \boldsymbol{F}_i^+ C_n^+ = \boldsymbol{F}_i^+ \frac{\Delta_{n1}^-}{\Delta_n} s_n = \begin{bmatrix} a_n^+ \\ b_n^+ \end{bmatrix}_{(i)}^{(L)} \frac{\Delta_{n1}^-}{\Delta_n} s_n. \quad (41)$$

Similarly to the case $i = j$, we have

$$(A_n^{(i)} \cosh \alpha_n y + B_n^{(i)} \sinh \alpha_n y) = (a_n^{i+} \cosh \overline{y} + b_n^{i+} \sinh \alpha_n \overline{y}) \frac{\Delta_{n1}^-(\overline{y_0})}{\Delta_n} s_n = \frac{\Delta_n^{i+}(\overline{h_i} - \overline{y})\Delta_n^{j-}(\overline{y_0})}{\Delta_n} s_n.$$

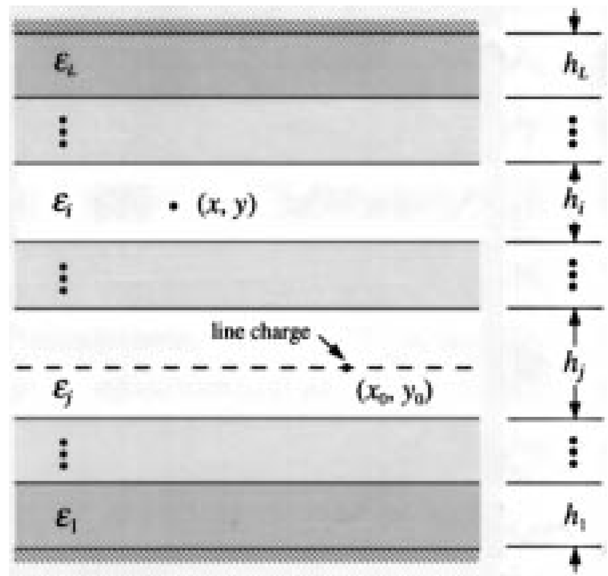

Fig. 3. An open multilayered medium structure.

Using the symmetry of the Green's function $i < j$ and combining the expression in (40), we can then unify the Green's functions at arbitrary observation point and source point in arbitrary layer as the formula given in (4).

IV. Analytical Green's Functions for an Open Multilayered Media

In an open structure as illustrated in Fig. 3, the Green's function [as that in (3)] can be expressed in the form of Fourier integral and given by

$$G_{ij}(x, y|x_0, y_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (A^{(i)} \cosh \gamma y + B^{(i)} \sinh \gamma y) \cdot \exp(-j\gamma x)\, d\gamma. \quad (42)$$

Similarly to the shielded structure, we have no difficulty in finding that

$$s = \frac{1}{\gamma} \exp(j\gamma x_0) \quad (43)$$

as shown in (22), and the Green's function can then be rewritten as

$$G_{ij}(x, y|x_0, y_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{2\gamma} \tilde{G}_{ij}(\gamma, y, y_0) \cdot \exp[-j\gamma(x - x_0)]\, d\gamma \quad (44)$$

where $\tilde{G}_{ij}(\gamma, y, y_0)$ is a spectral kernel function of the Green's function in y direction. The $\tilde{G}_{ij}(\gamma, y, y_0)$ can be obtained as an extension of the shielded structure by taking limits as $a \to \infty, n\pi/a \to \gamma$ in (4) and given by

$$\tilde{G}_{ij}(\gamma, y, y_0) = \tilde{G}_{ij}(\overline{y}, \overline{y_0}) = \frac{2}{\tilde{\Delta}} \begin{cases} \Delta^{i+}(\overline{h_i} - \overline{y})\Delta^{j-}(\overline{y_0}), & i > j \text{ or } i = j, y \ge y_0 \\ \Delta^{i-}(\overline{y})\Delta^{j+}(\overline{h_j} - \overline{y_0}), & i < j \text{ or } i = j, y \le y_0 \end{cases} \quad (45)$$

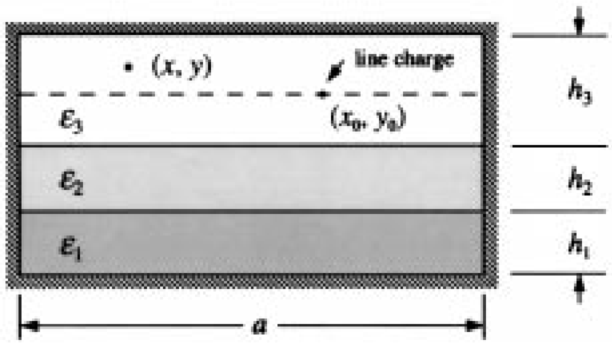

Fig. 4. A three-layered dielectric structure shielded with conductor walls.

where

$$\overline{h_k} = \gamma h_k, \qquad k = 1, 2, \cdots, L \tag{46a}$$

$$\overline{y} = \gamma y, \qquad \overline{y_0} = \gamma y_0 \tag{46b}$$

$$\tilde{\Delta} = \Delta(\overline{h_1}, \overline{h_2}, \cdots, \overline{h_L}) \tag{47a}$$

$$\Delta^{i+}(\overline{y}) = \Delta(0, 0, \cdots, \overline{y}, \overline{h_{i+1}}, \cdots, \overline{h_L}) \tag{47b}$$

$$\Delta^{i-}(\overline{y}) = \Delta(\overline{h_1}, \overline{h_2}, \cdots, \overline{h_{i-1}}, \overline{y}, 0, \cdots, 0). \tag{47c}$$

Note the similarity between the formula in (4) and that in (45). These two formulas have almost the same form except of the difference of the summation and the integration.

To simulate the true open structure in the vertical direction for both shielded and open structures, we can take the top layer to have infinite thickness. Taking the thickness of the top layer to be infinite yields two infinite hyperbolic sine or cosine functions for the layer that occur in the nominator and denominator of the Green's function at the same time, hence, cancelling the infinite exponential part of each other; therefore, this operation does not bring any difficulty in calculation. Similarly, the bottom layer can be also taken to be infinite, but at least one of the top or the bottom layers must not be infinite because the potential in the 2-D problem needs a reference potential or a ground at a finite distance.

V. Green's Functions for a Three-Layered Dielectric Structure

For a three-layered dielectric structure, as shown in Fig. 4, here we derive its Green's functions as an example of the application of the general analytical formula in (4).

Letting $L = 3$ into (4)–(7), we have the generating formula as

$$\begin{aligned}\Delta(z_1, z_2, z_3) = \frac{1}{\varepsilon_1\varepsilon_2\varepsilon_3}[&\varepsilon_2\varepsilon_2 \sinh z_1 \sinh z_2 \sinh z_3 \\ &+ \varepsilon_2\varepsilon_3 \sinh z_1 \cosh z_2 \cosh z_3 \\ &+ \varepsilon_3\varepsilon_1 \cosh z_1 \sinh z_2 \cosh z_3 \\ &+ \varepsilon_1\varepsilon_2 \cosh z_1 \cosh z_2 \sinh z_3]\end{aligned} \tag{48}$$

and then

$$\Delta_n = \Delta(\alpha_n h_1, \alpha_n h_2, \alpha_n h_3). \tag{49}$$

Placing the observation point and source point into each layer alternately gives

$$\Delta_n^{3+}(\overline{y}) = \frac{1}{\varepsilon_3} \sinh \alpha_n y \tag{50a}$$

$$\Delta_n^{3-}(\overline{y}) = \Delta(\alpha_n h_1, \alpha_n h_2, \alpha_n y) \tag{50b}$$

$$\begin{aligned}\Delta_n^{2+}(\overline{y}) = \frac{1}{\varepsilon_2\varepsilon_3}[&\varepsilon_3 \sinh \alpha_n y \cosh \alpha_n h_3 \\ &+ \varepsilon_2 \cosh \alpha_n y \sinh \alpha_n h_3]\end{aligned} \tag{51a}$$

$$\begin{aligned}\Delta_n^{2-}(\overline{y}) = \frac{1}{\varepsilon_1\varepsilon_2}[&\varepsilon_2 \sinh \alpha_n h_1 \cosh \alpha_n y \\ &+ \varepsilon_1 \cosh \alpha_n h_1 \sinh \alpha_n y]\end{aligned} \tag{51b}$$

$$\Delta_n^{1+}(\overline{y}) = \Delta(\alpha_n y, \alpha_n h_2, \alpha_n h_3) \tag{52a}$$

$$\Delta_n^{1-}(\overline{y}) = \frac{1}{\varepsilon_1} \sinh \alpha_n y \tag{52b}$$

and the Green's functions for each layer are then given as

$$\begin{aligned}&G_{33}(x, y|x_0, y_0) \\ &= \sum_{n=1}^{\infty} \frac{1}{\alpha_n a} \sin \alpha_n x \sin \alpha_n x_0 \\ &\cdot \frac{2}{\Delta_n} \begin{cases} \Delta_n^{3+}(\overline{h_3} - \overline{y})\Delta_n^{3-}(\overline{y_0}), & y \geq y_0 \\ \Delta_n^{3-}(\overline{y})\Delta_n^{3+}(\overline{h_3} - \overline{y_0}), & y \leq y_0 \end{cases}\end{aligned} \tag{53a}$$

$$\begin{aligned}&G_{32}(x, y|x_0, y_0) \\ &= \sum_{n=1}^{\infty} \frac{1}{\alpha_n a} \sin \alpha_n x \sin \alpha_n x_0 \cdot \frac{2\Delta_n^{3+}(\overline{h_3} - \overline{y})\Delta_n^{2-}(\overline{y_0})}{\Delta_n}\end{aligned} \tag{53b}$$

$$\begin{aligned}&G_{31}(x, y|x_0, y_0) \\ &= \sum_{n=1}^{\infty} \frac{1}{\alpha_n a} \sin \alpha_n x \sin \alpha_n x_0 \cdot \frac{2\Delta_n^{3+}(\overline{h_3} - \overline{y})\Delta_n^{1-}(\overline{y_0})}{\Delta_n}\end{aligned} \tag{53c}$$

$$\begin{aligned}&G_{22}(x, y|x_0, y_0) \\ &= \sum_{n=1}^{\infty} \frac{1}{\alpha_n a} \sin \alpha_n x \sin \alpha_n x_0 \\ &\cdot \frac{2}{\Delta_n} \begin{cases} \Delta_n^{2+}(\overline{h_2} - \overline{y})\Delta_n^{2-}(\overline{y_0}), & y \geq y_0 \\ \Delta_n^{2-}(\overline{y})\Delta_n^{2+}(\overline{h_2} - \overline{y_0}), & y \leq y_0 \end{cases}\end{aligned} \tag{54a}$$

$$\begin{aligned}&G_{21}(x, y|x_0, y_0) \\ &= \sum_{n=1}^{\infty} \frac{1}{\alpha_n a} \sin \alpha_n x \sin \alpha_n x_0 \cdot \frac{2\Delta_n^{2+}(\overline{h_2} - \overline{y})\Delta_n^{1-}(\overline{y_0})}{\Delta_n}\end{aligned} \tag{54b}$$

$$\begin{aligned}&G_{11}(x, y|x_0, y_0) \\ &= \sum_{n=1}^{\infty} \frac{1}{\alpha_n a} \sin \alpha_n x \sin \alpha_n x_0 \\ &\cdot \frac{2}{\Delta_n} \begin{cases} \Delta_n^{1+}(\overline{h_1} - \overline{y})\Delta_n^{1-}(\overline{y_0}), & y \geq y_0 \\ \Delta_n^{1-}(\overline{y})\Delta_n^{1+}(\overline{h_1} - \overline{y_0}), & y \leq y_0. \end{cases}\end{aligned} \tag{55}$$

The remaining Green's functions can be found from the symmetry of the Green's function and given by

$$G_{23}(x, y|x_0, y_0) = G_{32}(x_0, y_0|x, y) \tag{56a}$$

$$G_{12}(x, y|x_0, y_0) = G_{21}(x_0, y_0|x, y) \tag{56b}$$

$$G_{13}(x, y|x_0, y_0) = G_{31}(x_0, y_0|x, y). \tag{56c}$$

These Green's functions for a three-layered dielectric structure are the same as those functions available in many published papers and books, for example, in [5], [13], and [14]. This example also provides a direct verification of the correctness of the general formulas.

VI. Conclusion

We have derived the general analytical solutions of the static Green's functions for shielded and open arbitrarily multilayered medium structures. The Green's functions for a three-layered dielectric structure are also provided as an example of the application of the general expressions. We can say that these general analytical formulas will end the history of the exhaustive derivation of the Green's function for the multilayered medium structures. One obvious merit of these formulas is that they are given finally in analytical expressions and the expressions are true for arbitrary number of layers. Being analytical and applicable to arbitrary number of layers will be very helpful to develop efficient calculation techniques of the Green's function based on those expressions, for example, to extract the singular part from the Green's function, to introduce an approximate formula that may be more suitable to the numerical computation, and to develop a general computation program for an arbitrarily multilayered medium structure. On the other hand, applying the Green's functions to the p-BEM will give an effective technique for the analysis of the multilayered medium structures, which may include arbitrary cross-sectional dielectric substrates and strip conductors [5]. Furthermore, using the concept of the complex image charges will greatly improve the convergence of the Green's functions expressed in the Fourier integral [15], [16]. These analytical formulas are able to be extended to the solutions for three-dimensional static Green's functions, since the derivation procedure after expanding the Green's function in a 2-D plane is completely the same as that given in this paper. We believe that these general analytical formulas for Green's function presented in this paper will be a key to open the door of the analysis of the arbitrarily multilayered medium structures.

References

[1] R. E. Collin, *Field Theory of Guided Waves*, 2nd ed. New York: IEEE Press, 1991, ch. 2.

[2] C. A. Brebbia, *The Boundary Element Method for Engineers*. London: Petch, 1978.

[3] K. Li and Y. Fujii, "Indirect boundary element method applied to generalized microstripline analysis with applications to side-proximity effect in MMIC's," *IEEE Trans. Microwave Theory Tech.*, vol. 40, pp. 237–244, Feb. 1992.

[4] W. Schroeder and I. Wolff, "A new hybrid mode boundary integral method for analysis of MMIC waveguides with complicated cross-section," in *IEEE MTT-S Int. Microwave Symp. Dig.*, June 1989, pp. 711–714.

[5] K. Atsuki and K. Li, "Partial-boundary element method for analysis of striplines with arbitrary cross-sectional dielectric in multilayered media," *IEEE Trans. Microwave Theory Tech.*, vol. 43, pp. 1153–1161, May 1995.

[6] T. Itoh, "Generalized spectral domain method for multiconductor printed lines and its application to turnable suspended microstrip," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-26, pp. 983–987, Dec. 1978.

[7] ——, "Spectral domain immitance approach for dispersion characteristics of generalized printed transmission lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-28, pp. 733–736, July 1980.

[8] N. K. Das and D. M. Pozar, "A generalized spectral-domain Green's function for multilayer dielectric substrates with application to multilayer transmission lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-35, pp. 326–335, Mar.. 1987.

[9] R. Crampagne, M. Ahmadpanah, and J. L. Guiraud, "A simple method for determining the Green's function for a large class of MIC lines having multilayered dielectric substrates," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-26, pp. 82–87, Feb. 1978.

[10] W. Delbare and D. D. Zetter, "Space-domain Green's function approach to the capacitance calculation of multiconductor lines in multilayered dielectrics with improved surface charge modeling," *IEEE Trans. Microwave Theory Tech.*, vol. 37, pp. 1562–1568, Oct. 1989.

[11] H. Ogawa, T. Hasegawa, S. Banba, and H. Nakamoto, "MMIC transmission lines for multilayered MMIC's," in *IEEE MTT-S Int. Microwave Symp. Dig.*, Boston, MA, June 10–14, 1990.

[12] C. Wei, R. F. Harrington, J. R. Mautz, and T. K. Sarkar, "Multiconductor transmission lines in multilayered dielectric media," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-32, pp. 439–450, Apr. 1984.

[13] E. Yamashita and K. Atsuki, "Analysis of thick-strip transmission lines," *IEEE Trans. Microwave Tech.*, vol. MTT-19, pp. 120–122, 1971.

[14] F. Olyslager, N. Fach, and D. De Zutter, "New fast and accurate line parameter calculation of general multiconductor transmission lines in multilayered media," *IEEE Trans. Microwave Theory Tech.*, vol. 39, pp. 901–909, June 1991.

[15] Y. L. Chow, J. J. Yang, and G. E. Howard, "Complex images for electrostatic field computation in multilayered dielectric media," *IEEE Trans. Microwave Tech.*, vol. 39, pp. 1120–1125, July 1991.

[16] G. E. Howard, J. J. Yang, and Y. L. Chow, "A multipipe model of general strip transmission lines for rapid convergence of integral equation singularities," *IEEE Trans. Microwave Theory Tech.*, vol. 40, pp. 628–636, Apr. 1992.

Keren Li (M'93) was born in Jiangsu Province, China, on June 22, 1963. He received the B.Eng. degree (electronic engineering) from the Nanjing Institute of Technology (now Southeast University), Nanjing, China, in 1983, the M.Eng. degree (electronic engineering) from the University of Electrocommunications, Tokyo, Japan, in March 1987, and the Ph.D. degree (optical communications) from the University of Tokyo, Japan, in March 1991.

He became a Research Associate of the University of Electrocommunications, Tokyo, Japan, in 1991, and a Lecturer in 1994. His research interests include electromagnetic analysis, microwave circuits, optical waveguides and optical devices in high-speed optical communications, and the interactions between microwaves and optic waves.

Dr. Li is a member of the Institute of Electronics Information and Communication Engineers (IEICE) and the Institute of Electrical Engineers, both of Japan.

Kazuhiko Atsuki (M'89) was born in Tokyo, Japan, on November 2, 1942. He received the B.Eng. and M.Eng. degrees from the University of Electrocommunications, Tokyo, Japan, and the Dr.Eng. degree from the University of Tokyo, Japan, all in electrical engineering, in 1965, 1967, and 1979, respectively.

He became a Research Assistant in 1967, an Associate Professor in 1982, and a Professor in 1990, all in the Department of Electronics, University of Electrocommunications, Tokyo, Japan. Presently, his research deals with electromagnetic theory, microwave and optical waveguides, and its applications.

Dr. Atsuki is a member of the Institute of Electronics, Information and Communication Engineers (IEICE) and the Institute of Electrical Engineers, both of Japan.

Tsuyoshi Hasegawa (S'95) was born in Chiba, Japan, on January 12, 1968. He received the B.Eng. and M. Eng. degrees in electrical engineering from the University of Electrocommunications, Tokyo, Japan, in 1991 and 1993, respectively. He is currently working toward the Ph.D. degree at the same university.

His research interests include analysis of microwave circuits.